A MULTIPOINT METHOD OF THIRD ORDER

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Abstract

Let F be a mapping of the Banach space X into itself. A convergence theorem for the iterative solution of $F(x) = 0$ is proved for the multipoint algorithm $x_{n+1} = x_n - \phi(x_n)$ where $\phi(x) = F_x'^{-1}[F(x) + F(x - F_x'^{-1}F(x))]$ and F_x' is the Frechet derivative of F. The theorem guarantees that, under appropriate conditions on F, the multipoint sequence $\{x_n\}$ generated by ϕ converges cubically to a zero of F. The algorithm is applied to the non-linear Chandrasekhar integral equation

$$\frac{\omega_o}{2}x(t)\int_o^1 \frac{tx(s)}{s+t}ds - x(t) + 1 = 0$$

where $\omega_o > 0$. A discretization of the equations of iteration is discussed and some numerical results are given.

1. Introduction

Let F be a mapping of the Banach space X into itself. Considerable effort has been devoted to the study of iterative methods for the determination of zeros of F i.e. solutions of $F(x) = 0$ (see, for example, [1], [2], [3], [6], [7] and [8]). Here, we consider a multipoint method of third order for the solution of $F(x) = 0$ which is based on the algorithm

$$x_{n+1} = x_n - \phi(x_n) \tag{1}$$

where

$$\phi(x) = F_x'^{-1}[F(x) + F(x-F_x'^{-1}F(x))] \tag{2}$$

and F_x' is the Frechet derivative of F. This algorithm requires several evaluations of F and a single inversion of the linear operator F_x' at each iteration step. We show that, under appropriate conditions on F, the algorithm generates a sequence which converges cubically to a zero of F. Since most higher order methods require the computation of correspondingly high order Frechet derivatives, we feel that the algorithm (1) and its generalizations ([7]) have definite practical utility.

We apply the algorithm and convergence theorem to the Chandrasekhar integral equation

$$\frac{\omega_o}{2}x(t)\int_o^1 \frac{tx(s)}{s+t}ds - x(t) + 1 = 0 \tag{3}$$

where $\omega_o > 0$. The integral equation (3) is equivalent to the operator equation in $\mathscr{C}([0,1])$

$$F(x) = B(x,x) - Ix + 1 = 0 \tag{4}$$

where

$$B(x,x)(t) = \frac{\omega_o}{2} x(t) \int_o^1 \frac{t x(s)}{s+t} ds \tag{5}$$

and $\omega_o > 0$.

2. A Convergence Theorem

We now state and prove a convergence theorem for the algorithm (1). The result is similar to Kantorovich's theorem on the convergence of Newton's method ([3]).

THEOREM. *Suppose that* F *is twice continuously differentiable on the closed sphere* $\bar{S}(x_o, r) = \bar{S}$ *and that there are constants* B_o, d_o *and* K *such that*

(i) $F'^{-1}_{x_o}$ *exists and* $\|F'^{-1}_{x_o}\| \leq B_o$;

(ii) $\|F'^{-1}_{x_o}\| \, \|F(x_o)\| \leq d_o$;

(iii) $\sup\limits_{x \in \bar{S}} \{\|F''_x\|\} \leq K$;

(iv) $\|\phi(x_o)\| \leq d_o\left(1 + \frac{KB_o d_o}{2}\right)$;

(v) $\frac{81}{17}\eta_o \leq r$ *and* $B_o K \eta_o \leq 5/9$

where $\eta_o = d_o\left(1 + \frac{KB_o d_o}{2}\right)$. *Then the multipoint sequence* $\{x_n\}$ *for* F *based on* x_o (*i.e. the sequence* $x_n = x_{n-1} - \phi(x_{n-1})$) *converges* cubically to a root x^* of $F(x) = 0$ in $\overline{S}$ and the rate of convergence is given by

$$\|x_n - x^*\| \leq (\frac{81}{17})(\frac{8}{9})^{2n}(\frac{16}{9}h_o)^{3^{n-1}}\eta_o, \quad n = 1,2,\ldots \tag{6}$$

where $h_o = B_o K \eta_o$.

Proof: The proof is by induction on n. We shall define sequences B_n, d_n, η_n and h_n by setting

$$B_n = B_{n-1}/(1-h_{n-1}) \tag{7}$$

$$d_n = h^2_{n-1}\eta_{n-1}/2(1-h_{n-1}) \tag{8}$$

$$\eta_n = d_n(1 + \frac{KB_n d_n}{2}) \tag{9}$$

$$h_n = KB_n\eta_n \tag{10}$$

and we shall show that the basic hypotheses (i) - (v) are satisfied for x_n relative to the constants B_n, d_n and K. We then estimate $\|x_{n+m} - x_n\|$ and show that $\{x_n\}$ is a Cauchy sequence in $\overline{S}$. Since $\overline{S}$ is complete, $\{x_n\}$ converges to an element x^* of $\overline{S}$ which we prove is a zero of F.

We begin with the transition from $n = 0$ to $n = 1$. Since F is twice differentiable on $\overline{S}$,

$$\|F'_x - F'_y\| \leq K\|x-y\| \tag{11}$$

for all x,y in $\overline{S}$ and so, in particular, we have

$$\|F'_{x_o}\| - \|F'_{x_1}\| \leqq \|F'_{x_o} - F'_{x_1}\| \leqq K\|x_1 - x_o\|$$

[note that $\|x_1 - x_o\| = \|\phi(x_o)\| \leqq \eta_o \leqq r$]. It follows that

$$\|F'_{x_1}\| \geqq [1 - \frac{K\eta_o}{\|F'_{x_o}\|}]\|F'_{x_o}\| \geqq (1-h_o)\|F'_{x_o}\| > 0$$

and hence that ${F'_{x_1}}^{-1}$ exists. Moreover, $\|{F'_{x_1}}^{-1}\| - \|{F'_{x_o}}^{-1}\| \leqq B_o h_o/(1-h_o)$ so that

$$\|{F'_{x_1}}^{-1}\| \leqq B_o/(1-h_o) = B_1. \tag{12}$$

Letting $y_o = x_o - {F'_{x_o}}^{-1}F(x_o)$ and noting that $\|y_o - x_o\| = \|{F'_{x_o}}^{-1}F(x_o)\| \leqq d_o \leqq \eta_o \leqq r$, we expand $F(x_1)$ in a Taylor series about y_o to obtain the inequality

$$\|F(x_1) - F(y_o) + F'_{y_o}{F'_{x_o}}^{-1}F(y_o)\| \leqq \frac{K}{2}\|{F'_{x_o}}^{-1}\|^2\|F(y_o)\|^2$$

which, in view of (11), yields the inequality

$$\|F(x_1)\| \leqq K\|{F'_{x_o}}^{-1}\|^2\|F(y_o)\|\{\|F(x_o)\| + \frac{\|F(y_o)\|}{2}\}.$$

Since $\|F(y_o) - F(x_o) - F'_{x_o}(-F'^{-1}_{x_o}F(x_o))\| \leq \frac{K}{2}\|F'^{-1}_{x_o}\|^2\|F(x_o)\|^2$ implies that $\|F(y_o)\| \leq \frac{K}{2}d_o^2$, we deduce that $\|F(x_1)\| \leq \frac{K^2d_o^2B_o}{2}(d_o+\frac{Kd_o^2B_o}{4})$ and hence that

$$\|F'^{-1}_{x_1}\| \ \|F(x_1)\| \leq \frac{h_o^2\eta_o}{2(1-h_o)} = d_1. \tag{13}$$

We now observe that $\eta_1 \leq (\frac{701}{512})h_o^2\eta_o \leq (\frac{5}{9})^2(\frac{701}{512})\eta_o \leq \eta_o$ so that $\eta_1 + \eta_o \leq \{(\frac{5}{9})^2(\frac{701}{512}) + 1\}\eta_o \leq \frac{81}{17}\eta_o \leq r$. We claim that $x_2 = x_1 - F'^{-1}_{x_1}[F(x_1) + F(x_1-F'^{-1}_{x_1}F(x_1))]$ is a well-defined element of $\overline{S}$ and that $\|x_2-x_1\| \leq \eta_1$. Letting $y_1 = x_1 - F'^{-1}_{x_1}F(x_1)$ and noting that $\|y_1-x_1\| = \|F'^{-1}_{x_1}F(x_1)\| \leq d_1$ so that $y_1 \in \overline{S}$, we can see that x_2 is defined and that $\|x_2-x_1\| = \|\phi(x_1)\|$. Now, $\|\phi(x_1)\| \leq \|F'^{-1}_{x_1}F(x_1)\| + \|F'^{-1}_{x_1}F(y_1)\|$ and $\|F(y_1) - F(x_1) - F'_{x_1}(-F'^{-1}_{x_1}F(x_1))\| \leq \frac{K}{2}\|F'^{-1}_{x_1}\|^2\|F(x_1)\|^2$, together imply that

$$\|\phi(x_1)\| \leq d_1(1 + \frac{KB_1d_1}{2}) = \eta_1. \tag{14}$$

Since $\frac{81}{17}\eta_1 \leq \frac{81}{17}\eta_o$ and since $h_1 = KB_1\eta_1$ so that

$$h_1 \leq \frac{KB_o}{(1-h_o)}(\frac{701}{512})h_o^2\eta_o \leq (\frac{701}{512})\frac{h_o^3}{1-h_o} \leq (\frac{80}{81})^2h_o \leq h_o \leq \frac{5}{9},$$

we deduce that the basic hypotheses (i) - (v) are satisfied for x_1 relative to the constants B_1, K and d_1.

We can show, by exactly the same arguments, that if the basic hypotheses (i) - (v) are satisfied for x_n relative to B_n, K and d_n, then the same holds true for x_{n+1} relative to B_{n+1}, K and d_{n+1}. Moreover, we have

$$\eta_{n+1} \leqq \left(\frac{701}{512}\right) h_n^2 \eta_n \tag{16}$$

and

$$h_{n+1} \leqq \left(\frac{16}{9}\right)^2 h_n^3 \tag{17}$$

for $n = 0, 1, 2, \ldots$. It follows that

$$h_n \leqq \left(\frac{9}{16}\right)\left(\frac{16}{9} h_o\right)^{3^n} \tag{18}$$

and

$$\eta_n \leqq \left(\frac{8}{9}\right)^{2n} \left(\frac{16}{9} h_o\right)^{3^{n-1}} \eta_o^{\dagger} \tag{19}$$

for $n = 1, 2, \ldots$. Since the x_n are in $\overline{S}$ and since the series

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†This follows from (16) and (18) as

$$\eta_n \leqq \left(\frac{701}{512}\right)^n \left(\frac{9}{16}\right)^{2n} \left(\frac{16}{9} h_o\right)^{\left(\sum_{k=1}^{n} 3^{n-k}\right) \times 2} \eta_o \leqq \left(\frac{701}{512} \times \frac{9}{16}\right)^n \left(\frac{16}{9} h_o\right)^{3^{n-1}} \eta_o$$

$$\leqq \left(\frac{8}{9}\right)^{2n} \left(\frac{16}{9} h_o\right)^{3^{n-1}} \eta_o .$$

$\sum \eta_n$ is convergent and since

$$\|x_{n+m}-x_n\| \leqq \sum_{o}^{m-1} \eta_{n+k} \leqq \sum_{o}^{m-1} (\frac{8}{9})^{2(n+k)} (\frac{16}{9}h_o)^{3^{n+k}-1} \eta_o$$

$$\leqq (\frac{8}{9})^{2n} (\frac{16}{9}h_o)^{3^{n-1}} \eta_o \cdot \sum_{o}^{m-1} (\frac{8}{9})^{2k} \tag{20}$$

$$\leqq (\frac{81}{17}) (\frac{8}{9})^{2n} (\frac{16}{9}h_o)^{3^{n-1}} \eta_o$$

we deduce that $\{x_n\}$ is a Cauchy sequence in $\bar{S}$ and, therefore, has a limit x^* in $\bar{S}$ with

$$\|x^*-x_n\| \leqq (\frac{81}{17}) (\frac{8}{9})^{2n} (\frac{16}{9}h_o)^{3^{n-1}} \eta_o \tag{21}$$

for $n = 0, 1,2,\ldots$.

It remains to show that x^* is a zero of F. Now, the induction argument yields (compare (13)) the inequality

$$\|F(x_n)\| \leqq \frac{h_{n-1}^2 \eta_{n-1}}{2(1-h_{n-1})B_n} \leqq \frac{h_{n-1}^2 \eta_{n-1}}{2B_{n-1}} \leqq \frac{h_{n-1}^2 \eta_{n-1}}{2B_o} \tag{22}$$

for $n = 1,2,\ldots$, since $B_{n-1} \geqq B_o$. Since F is continuous, we deduce that x^* is a zero of F from (18) and (19). Thus, the theorem is established.

We note that the numerical estimates in condition (v) and in equation (6) are not quite optimum but rather are chosen for convenience of calculation.

3. A Nonlinear Integral Equation

We consider the nonlinear Chandrasekhar integral equation (3). We let X be the Banach space of continuous functions on $[0,1]$, $\mathscr{C}([0,1])$, under the uniform (or supremum) norm and we let $B(\cdot,\cdot)$ be the bilinear map of $X \oplus X$ into X given by

$$B(u,v)(t) = \frac{\omega_o}{2} u(t) \int_o^1 \frac{t}{s+t} v(s)ds \tag{23}$$

for $t \in [0,1]$. Then determining a solution of (3) is equivalent to determining the roots of the "quadratic" operator equation

$$F(x) = B(x,x) - Ix + 1 = 0 \tag{24}$$

in $\mathscr{C}([0,1])$.

Now we observe that the operator F is twice continuously Frechet differentiable and that

$$F'_x z = B(x,z) + B(z,x) - Iz \tag{25}$$

$$F''_x(w,z) = B(w,z) + B(z,w) \tag{26}$$

for any x in $\mathscr{C}([0,1])$. We then have:

COROLLARY. <u>Suppose that</u> $0 < \omega_o \leq .65$ <u>and that</u> $r \geq \frac{81}{17}(1+\omega_o \ell n2)$. <u>The the multipoint sequence</u> $\{x_n\}$ <u>generated by the algorithm</u> (1)

with $x_o = 0(\cdot)$ converges to a solution x^* of (3) in $\overline{S}(0,r)$ and the rate of convergence is given by (6) with $\eta_o = (1+\frac{\omega_o \ell n2}{2})$ and $h_o = \omega_o \ell n2(1+\frac{\omega_o \ell n2}{2})$.

Proof: We simply show that the hypotheses of the theorem are satisfied. Since $x_o = 0(\cdot)$, we have $F'_{x_o} = -I$ and $F(x_o) = 1(\cdot)$ by virtue of (25) and the definition of F. Thus, $\|F'^{-1}_{x_o}\| = 1$ and $\|F'^{-1}_{x_o}\| \, \|F(x_o)\| = 1$ so that we may take $B_o = d_o = 1$ and the hypotheses (i) and (ii) will be satisfied. As for (iii), we note that

$$|F''_x(w,z)(t)| \leq \frac{\omega_o}{2}\{|w(t)|\int_o^1 \frac{t}{s+t}|z(s)|ds + |z(t)|\int_o^1 \frac{t}{s+t}|w(s)|ds\}$$

$$\leq \omega_o \max_{0\leq t\leq 1} \{\int_o^1 \frac{t}{s+t}ds\}\cdot\|w(\cdot)\|\cdot\|z(\cdot)\|$$

$$\leq (\omega_o \ell n2)\|w(\cdot)\| \, \|z(\cdot)\|$$

which implies that $\|F''_x\| \leq K$ for all x with $K = \omega_o \ell n2$. Now $\phi(x_o) = -I[1(\cdot) + F(-1(\cdot))] = -1(\cdot) - B(-1(\cdot)) + 1(\cdot) + 1(\cdot) = 1(\cdot) - B(-1(\cdot), -1(\cdot))$ and $B(-1(\cdot), -1(\cdot))(t) = \frac{\omega_o}{2}\int_o^1 \frac{t}{s+t}ds$ so that $\|\phi(x_o)\| \leq 1 + \frac{\omega_o \ell n2}{2} = \eta_o$ and the hypothesis (iv) is satisfied. The hypothesis (v) is satisfied by virtue of the assumptions of the corollary. Thus the corollary follows from the theorem.

We now describe the algorithm (1) for the mapping F of (24).

Setting $y_n = x_n - F'^{-1}_{x_n}F(x_n)$, we have

$$F'_{x_n}(x_{n+1}-x_n) = -F(x_n) - F(y_n) \tag{27}$$

and

$$F'_{x_n}(y_n-x_n) = F(x_n) \tag{28}$$

for $n = 1,2,\ldots$. Thus, the algorithm (1) is equivalent to the <u>system</u>

$$\begin{aligned} B(x_n,y_n) + B(y_n,x_n) - y_n &= B(x_n,x_n) - 1(\cdot) \\ B(x_n,x_{n+1}) + B(x_{n+1},x_n) - x_{n+1} &= B(x_n,x_n) - 1(\cdot) - F(y_n) \end{aligned} \tag{29}$$

which can be written more explicitly in the form

$$\begin{aligned} &\frac{\omega_o}{2}[x_n(t)\int_o^1 \frac{t}{s+t}y_n(s)ds + y_n(t)\int_o^1 \frac{t}{s+t}x_n(s)ds] - y_n(t) = \\ &\frac{\omega_o}{2}x_n(t)\int_o^1 \frac{t}{s+t}x_n(s)ds - 1 \end{aligned} \tag{30a}$$

$$\begin{aligned} &\frac{\omega_o}{2}[x_n(t)\int_o^1 \frac{t}{s+t}x_{n+1}(s)ds + x_{n+1}(t)\int_o^1 \frac{t}{s+t}x_n(s)ds] - x_{n+1}(t) = \\ &\frac{\omega_o}{2}[x_n(t)\int_o^1 \frac{t}{s+t}x_n(s)ds - y_n(t)\int_o^1 \frac{t}{s+t}y_n(s)ds] + y_n(t) - 2. \end{aligned} \tag{30b}$$

The system (30) is a pair of "similar" linear Fredholm equations for y_n and x_{n+1}, respectively. We shall see, in the next section, that only one of the pair of equations need be solved at each step in the "practical" application of the method.

4. Discretization and Numerical Results

The corollary quarantees the convergence of the algorithm (1) for the Chandrasekhar integral equation with $0 < \omega_o \leqq .65$. However, it is clear from the form of (30) that the pair of linear equations required at each iteration step is not easily solved. Thus, we shall "discretize" the system in an appropriate way and then carry out the calculations on a computer. The chief advantage of the multipoint method will then become apparent since only one "matrix inversion" will be required at each iteration step.

Now let $\Gamma(t,s) = t/(s+t)$ and let $t_o,\ldots,t_m$ be a partition (or mesh) on $[0,1]$ with $t_o = 0$ and $t_m = 1$. Suppose that $(\gamma_1,\ldots,\gamma_m,s_1,\ldots,s_m)$ represents a suitable quadrature rule with the γ's as weights and the s_i as mesh points in $[0,1]$. Then, we have

$$\int_o^1 \Gamma(t_i,s)x(s)ds \approx \sum_{j=1}^{m} \gamma_j \Gamma(t_i,s_j)x(s_j) \tag{31}$$

for $i = 0,1,\ldots,m$ and so,

$$\int_0^1 \Gamma(t,s)x(s)ds \approx \begin{pmatrix} \gamma_1\Gamma(t_1,s_1) & \cdots & \gamma_m\Gamma(t_1,s_m) \\ \vdots & & \\ \gamma_1\Gamma(t_m,s_1) & \cdots & \gamma_m\Gamma(t_m,s_m) \end{pmatrix} \begin{pmatrix} x(s_1) \\ \vdots \\ x(s_m) \end{pmatrix} \quad (32)$$

provides an m-dimensional representation of $\int_0^1 \Gamma(t,s)x(s)ds$. In a similar way, we can represent (approximate) the bilinear operator B by the m-dimensional bilinear operator B^m defined by the "three dimensional matrix", $\underline{B}^m$ with

$$\underline{B}^m = (b_{ijk}) = \frac{\omega_o}{2}(\gamma_j\Gamma(t_i,s_j)\delta_{ik}) \quad (33)$$

(δ_{ik} being the Kronecker delta). In other words, if $\underline{\alpha}$ and $\underline{\beta}$ are m vectors with components $\alpha_1,\ldots,\alpha_m$ and $\beta_1,\ldots,\beta_m$, respectively, then $\underline{B}^m(\underline{\alpha},\underline{\beta})$ is the m vector with components $\sum_{j=1}^m \sum_{k=1}^m b_{ijk}\alpha_k\beta_j$. Thus, if we set $\underline{x} = (x(t_1),\ldots,x(t_m))'$, then we can represent (24) in the discrete form

$$\underline{F}(\underline{x}) = \underline{B}^m(\underline{x},\underline{x}) - \underline{I}^m\underline{x} + \underline{1}^m = \underline{0} \quad (34)$$

where $\underline{I}^m$ is the $m \times m$ identity matrix and $\underline{1}^m$ is an m vector with all components equal to 1. Applying the algorithm (1) directly to (34), we obtain a system of equations identical to the discretized version of (30). In other words, we have

$$\underline{B}^m(\underline{x}_n,\underline{y}_n) + \underline{B}^m(\underline{y}_n,\underline{x}_n) - \underline{y}_n = \underline{B}^m(\underline{x}_n,\underline{x}_n) - \underline{1}^m \tag{35a}$$

$$\underline{B}^m(\underline{x}_n,\underline{x}_{n+1}) + \underline{B}^m(\underline{x}_{n+1},\underline{x}_n) - \underline{x}_{n+1} = \underline{B}^m(\underline{x}_n,\underline{x}_n) - \underline{1}^m - \underline{F}(\underline{y}_n) \tag{35b}$$

or, equivalently,

$$\underline{M}_n\underline{y}_n = \underline{c}_n \tag{36a}$$

$$\underline{M}_n\underline{x}_{n+1} = \underline{c}_n - \underline{\Delta}_n \tag{36b}$$

where $\underline{M}_n$ is the $m \times m$-matrix given by

$$\underline{M}_n = [\underline{B}^m(\underline{x}_n,\cdot) + \underline{B}^m(\cdot,\underline{x}_n) - \underline{I}^m] \tag{37}$$

and $\underline{c}_n$, $\underline{\Delta}_n$ are the m vectors given by

$$\underline{c}_n = \underline{B}^m(\underline{x}_n,\underline{x}_n) - \underline{1}^m,\ \underline{\Delta}_n = \underline{B}^m(\underline{y}_n,\underline{y}_n) - \underline{y}_n + \underline{1}^m. \tag{38}$$

Thus, if $\underline{M}_n$ is invertible, we have

$$\underline{y}_n = \underline{M}_n^{-1}\underline{c}_n \tag{39a}$$

$$\underline{x}_{n+1} = \underline{M}_n^{-1}(\underline{c}_n - \underline{\Delta}_n) \tag{39b}$$

and so only a single inversion is required at each iteration step.

Although the approximate solution is in "discrete form", we can fit an interpolating polynomial to the points $x_n(t_k)$ after the desired degree of accuracy has been obtained. It can also be shown (cf. [4]) that the operations of "discretization" and "iteration" are commutative and an error analysis, based on this fact, can be carried out.

Results of some particular computations appear in the following tables. The interval $[0,1]$ was partitioned into 10 subintervals of length $h = 0.1$ and Simpson's rule was used for the numerical integration. In Table I, we present the number of iterations required to obtain "convergence"† for various values of ω_o, and, in Table II, we present the actual approximate solutions. The results in Table II compare quite favorably with those presented by Rall in [5]. We also observe that, although the corollary guaranteed convergence only for $0 < \omega_o \leq .65$, the actual computations converged for values of $\omega_o > .65$.

TABLE I

ω_o	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Iterations Required	2	2	2	3	3	3	3	4	4	5

† Convergence is here construed to mean that

$$\|\underline{x}_{n+1}-\underline{x}_n\| \leq 1.0 \times 10^{-7}$$

where $\|\cdot\|$ is the Euclidean norm in R_{11}.

TABLE II

t \ ω_o	0.1	0.2	0.3	0.4	0.5
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.0124406	1.0257406	1.0400543	1.0555925	1.0726499
0.2	1.0186557	1.0389366	1.0611696	1.0858020	1.1134719
0.3	1.0230083	1.0482969	1.0763597	1.1078805	1.1438493
0.4	1.0263061	1.0554552	1.0880966	1.1251392	1.1679153
0.5	1.0289211	1.0611705	1.0975427	1.1391573	1.1876716
0.6	1.0310583	1.0658683	1.1053562	1.1508398	1.2042799
0.7	1.0328445	1.0698127	1.11195183	1.1607608	1.2184896
0.8	1.0343627	1.0731792	1.1176071	1.1693124	1.2308130
0.9	1.0356721	1.0760908	1.1225166	1.1767711	1.2416200
1.0	1.0368137	1.0786371	1.1268243	1.1833400	1.2411844

t \ ω_o	0.6	0.7	0.8	0.9	1.0
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.0916738	1.1133937	1.1391964	1.1725349	1.2441701
0.2	1.1451549	1.1824722	1.2285375	1.2912197	1.4416694
0.3	1.1857929	1.236290	1.3003721	1.3909588	1.6277685
0.4	1.2285030	1.280475	1.3608188	1.47799668	1.8075085
0.5	1.2456979	1.3177642	1.4129552	1.5554475	1.9830198
0.6	1.2688045	1.398840	1.4586525	1.6252145	2.1554050
0.7	1.2887496	1.3779287	1.4991760	1.6886072	2.3253097
0.8	1.3061819	1.4026842	1.5354414	1.7465906	2.4931469
0.9	1.3215723	1.4247322	1.5681352	1.7999086	2.6591930
1.0	1.3352737	1.4445133	1.5977897	1.8491525	2.823638

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